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FUNCTIONAL AND MATHEMATICAL EQUIVALENCE OF MECHANISMS: A NOVEL APPROACH TO INTEGRATING SYNTHESIS AND DESIGN ANALYSIS

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ABSTRACT

The objective of our research is to produce classical engineering design analysis models in conjunction with the employment of function based synthesis techniques. As a part of this goal, we define a design approach to construct parametric design models from functional models. Such an approach allows a natural progression from functional product models, to synthesis, to analysis, and then to initial parameter specification. An empirical study is performed to gather product data from a jigsaw and a palm sander. Our method starts by creating a black box model and a functional model of a product. From the functional model, a critical flow and an associated function chain are identified. The next step is to express each component at a functional level. Following that, components are mapped to equations describing their performance and finally, functions are mapped to equations. We analyze product data that is collected following the above described steps and discover functional and mathematical similarities between a scotch yoke mechanism in a jigsaw and an eccentric cam in a palm sander; in this case, two mechanisms used to perform the same function. We derive general performance equations for functions based on the components that are functionally similar but differing in form.

Keywords: Functional Modeling, Synthesis, Design Analysis, Design Models.

1. INTRODUCTION

Engineers design products and systems to fulfill some need or needs by performing some function or functions. Developing a method that enables the construction of parametric design models from functional models makes a fundamental contribution to engineering design practice, education, and research. Models that allow designers to synthesize and analyze simultaneously hold out the possibility of connecting customer needs, function, and performance of a product.

In this paper we use the term function to describe both overall product function and the sub-functions that are used in a decomposed and detailed description of overall product function. Whether overall product function or a specific sub-function is intended by the term function will be clear by context. In addition, we define parametric design models as an abstraction of combined mathematical equation(s) that describe the performance of some component or sub-system of a product. Synthesis is the assembly of parts or elements to produce new effects and to demonstrate that these effects create an overall order. Analysis is the resolution of anything complex into its elements and the study of these elements and their interrelationships (Pahl and Beitz, 1998). As a contribution to engineering design research, a method that integrates synthesis and analysis would be a step toward a quantified relationship between function and form. In this article, we present results of our efforts to develop a method that allows a much smaller discontinuity between synthesis and analysis than possible with current methods and knowledge. With such a result, engineering designers can use this function-to-parametric modeling design approach to effectively perform synthesis and analysis simultaneously.

The paper is organized as follows: Section 2 provides a review of related work. Section 3 gives an overview of the modeling methodology and our research approach. In Section 4, we employ the research method and explore the functional and mathematical equivalence of mechanisms in both jigsaw and palm sander. Finally conclusions are drawn and future work is discussed in Section 5.

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2. BACKGROUND AND RELATED WORK

Automated computation in the conceptual design stage, whether quantitatively evaluating solution concepts in an automated, semi-automated, or otherwise intelligent design-assisted manner, is a challenging and active research area. In function-based design methodologies, functional modeling of a device is a critical step in the design process (Pahl and Beitz, 1988; Suh, 1990). In response to the systematic approach of Pahl and Beitz and Hubka (1984), representing European schools of design, different methodologies were generated in American design literature (Ullman, 1997; Ulrich and Eppinger, 1995; Schmidt and Cagan, 1995; Pimmler and Eppinger, 1994; Shimomura et al., 1996; Cutherell, 1996; Otto and Wood, 1996, 2001). Regardless of the methodology variation, the basic premise of these methods is that once a product's functionality is clearly defined, architecture decisions (such as platforms or component reuse) and physical solution selections may be made more intelligently. These methods have been widely used to enhance creativity in design and decompose problems that are too complex solve as a whole.

Although there are general methodologies dealing with functions in design, virtually no commercial CAD systems can support functional based design methods, in particular the so called synthetic phase of design. Umeda (1996) has developed a computer tool called a Function-Behavior-State (FBS) Modeler that supports functional design in both the analytical phase and synthetic phase. The modeler is based on the specifications of required functions, functional decomposition, embodiment of functions, causal decomposition, construction of behavioral networks, and simulation of these networks and evaluation. The FBS modeler is in the experimental stage and has not yet been verified for application in actual design. Umeda has yet to devise an approach to evaluate the design objects quantitatively.

Chakrabarti (1996) devised an approach to functional synthesis of mechanical design concepts. Based on the premise that there are many solutions to a design problem, Chakrabarti proposed there is potential for producing improved designs if one can explore a solution space as large as possible. His approach uses a computer program to synthesize a wide variety of concepts for a given problem and allows designers to explore these before developing the most promising ones. The program produces exhaustive sets of solution concepts in terms of their topological and spatial configurations. In the study, he shows how certain devices and mechanisms are similar, such as a door latch, paper punch and a scotch yoke mechanism. Chakrabarti's program allows the definition of a design problem only in terms of its instantaneous characteristics of inputs and outputs. The result is that generated solutions are ensured to satisfy functions at only an instant in time.

Stein (1996) studied and developed a template based modeling approach. His modeling approach begins with engineers examining some real phenomena and then, based on engineering specifications, modeling decisions that account for system behavior are made to generate a model. This model is usually a collection of interconnected ideal elements for models of mechanical systems. The development of an appropriate physical model requires engineering intuition, judgment, and experience gained through extensive exposure to modeling of systems. This modeler is used only in linear dynamic systems and cannot be used for non-linear system models.

Design focused models have been developed by Roth (1987). Design models are developed at various stages of the product design process, making models more concrete at each stage by adding information. This systematic guide leads from an overall function to design sketches. It facilitates the design process and ensures that more solutions are considered and varied so that a superior solution can be selected. Performance modeling of these functional components is not emphasized.

In Sumida (1998) a modeling application to automobile development is described. A block diagram representation of a product, using main symbols and their meanings, was developed. These models are developed using a limited symbols and not sufficient for a designer to model systems from multiple domains.

In a recent effort, Bryant et. al. (2001) began the development of functional models and mathematical equations describing product functions. These equations are placed into function flow diagrams. Then, mathematical equations relating to performance parameters are extracted. This approach was used for various products and a partial handbook of equations was developed. The functional modeling methodology is based on the prior efforts of Stone and Wood (2000) and Hirtz et. al. (2002). The partial handbook of equations helps a designer see what equations govern a product's performance. The handbook of equations also points toward a mathematical similarity in different systems with the same function.

The research results presented in this article build upon, and extend, the above reviewed work. Here, functional modeling and mathematical models are combined to create the knowledge and process to benefit design and development efforts, engineering design education, and take a step toward a fundamental understanding of the design relationship between function and form.

3. RESEARCH APPROACH

Our research approach is based on the following question: Can a relationship between parametric design models (i.e. mathematical equations) and product function be established at an abstracted level in the conceptual design stage? In essence, we seek to integrate synthesis and analysis activities of design.

3.1 Modeling Philosophy

To integrate the activities of synthesis and analysis, we build upon function-based design methodologies to illuminate the link between function and form. Our specific approach is to build on the function model construction method presented by Bryant et. al. (2001). Through a discussion and explanation of Figure 1, we explain the philosophy and steps of our research approach.

Figure 1 illustrates both a functional approach to product design and the corresponding state of design modeling knowledge. In turn, we use this relationship between functional modeling and analytical modeling to motivate our present work and to develop a research approach to identify the relationship. In short, our goals are to develop methods and knowledge that allow the natural development of engineering design models at the earliest stages of design when only the desired function of a product or product subsystem is known. Moving down the left set of blocks in Figure

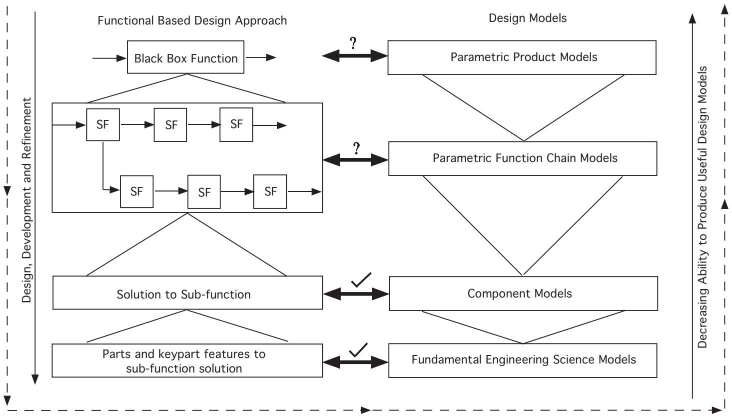


Figure 1. Functional approach to product design.

1, we look at four stages of a design development and synthesis. In the first stage the product is abstracted to a simple black box function. In the second stage, a complete functional model is developed. In the third stage, solution principles are explored and solution concepts for each function generated and selected. In the last stage, these solution concepts are embodied by specifying materials, geometry, and other parameter values.

On the right side of Figure 1, the engineering analysis models associated with the different stages of the design development are shown. The heavy double arrowhead line represents moving back and forth between the analysis and synthesis as the designer creates, checks, and refines proposed concepts and decisions. A check mark on the double arrow indicates that knowledge, science, and methods exist that enable designers to traverse this path between analysis and the associated synthesis effort. In the figure there are check marks on the bottom two arrows to indicate the existence and availability of this knowledge.

As an example of moving between synthesis and analysis along the bottom row of Figure 1, consider the design of a bicycle crank arm. To specify a specific geometry and material on a bicycle crank arm, an engineer can use methods from an undergraduate mechanics of materials course, or, if more resolution is needed in the analysis, a finite element model. As an example of moving between synthesis and analysis on the second row, a designer may be attempting to explore the specification or selection of an appropriate electric motor. Here, the designer can use higher-level system models to check appropriate power, torque, speed, and dynamics. Currently many techniques exist to approach this type of analysis effort.

Moving upward in the figure toward the earlier stages of design, the knowledge available to designers grows scarce. To represent this, the check mark is replaced by a question mark. For example, consider the design synthesis and analysis of a jigsaw. A designer might be considering different combinations of elements that convert the input electrical energy to the output oscillating mechanical energy. At the same time, issues of housing size are important. With the current state of knowledge, the designer is generally reduced to estimates of parameter values and the associated relationships. Conceptualizing and selecting primary system components is critical to successful design. What the designer needs to facilitate this synthesis effort is quick, cheap, and accurate models.

Moving up to the top row, we have the notion of complete parametric product models. Though such a model is difficult to conceptualize (e.g., what is the model of an automobile?), constructing complete product models allows us to approach the notion of a truly optimal design. With such models, customer needs would be used to construct an objective function, input flows would identify the design parameters of the system, and these parameters would be varied until customer satisfaction is optimal. Given the current state of the art, we are admittedly some distance from achieving such a goal.

3.2 Populating a Knowledge Base

Working through the ideas illustrated in Figure 1 highlights both critical areas of research need and our approach to addressing the needs. While our goal is to make the leap from function to mathematical equation at the concept synthesis phase, we first must have a knowledge base of the relationship between function and an associated mathematical equation. To populate the required knowledge base, our approach uses reverse engineering methods and is represented by the dashed line at the perimeter of Figure 1. Given some product, a functional model is constructed. As functions reduce to components and basic elements, design models are constructed; moving upward on the right hand side, these models are correlated with functions and combined to produce function chain models. With both a functional model and its associated function chain mathematical model in place, the double arrow between the functional and mathematical representation can be explored. This empirical approach follows the basic scientific method: observation, hypothesis, and test of hypothesis. Because of the complexity of the problem, and the need for a general solution, at this point the research presented here is largely focused on the creation of knowledge at the observation stage and an initial formulation and test of hypothesis. We are observing the connection between functional models and mathematical models. Thus, a key at this point in the research is to carefully record the results and determine how the results are produced. Below, our specific observation method is explained. An example of the method is given, and how the results can be used in a constructive design context is shown.

Following the philosophies presented above, our approach to populate the knowledge base is an eight- step process that is both empirical and constructive.

Step 1: Develop a black box and complete functional model of a product. This black box function of the product is developed from customer needs. A complete functional model of the product is created using active verb-object functional description and the functional basis format (Hirtz et al. 2002).

Step 2: Identify a flow of interest and the associated functions chain based on important customer needs. This is done by identifying important customer needs, correlating those customer needs to a flow, and following that flow through the functional model to identify a chain of functions. The resultant chain of functions represents a subset of product functionality that is of critical importance. In other words, a flow of interest is one that is important to overall customer satisfaction.

Step 3: Adapt function chains to model each component. A greater level of detail is added to the model through this step,

Function-Component Matrix

	Motor	Eccentric cam
Import rot.e	1	0
Convert rot.e to trans.e	0	1

X

Component-Equation Matrix

	$\tau = A - B\omega$	$\frac{\partial x}{\partial t} = e\sin\theta\frac{\partial\theta}{\partial t}$
Motor	1	0
Eccentric cam	0	1

=

Function-Equation Matrix

	$\tau = A - B\omega$	$\frac{\partial x}{\partial t} = e\sin\theta\frac{\partial\theta}{\partial t}$
Import rot.e	1	0
Convert rot.e to trans.e	0	1

Figure 2. Matrix multiplication.

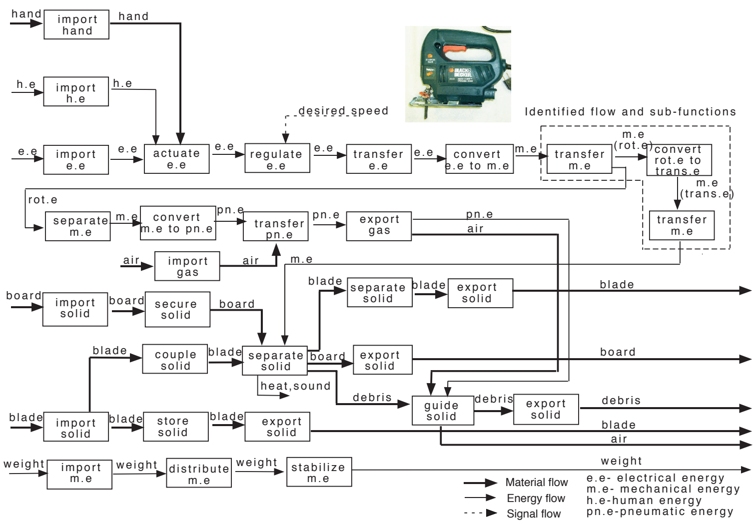


Figure 3. Functional model of a jigsaw.

uncovering and representing interfacial functionality of each component. At the conclusion of this step, the connection between function and each physical part of the product can be explicitly and unambiguously described (Modarres, 1999).

Step 4: Map each function to the component that solves it. This information is recorded in a Function-Component matrix (**FC**).

Step 5: Formulate design equations for each component. This information is recorded in a Component-Equation matrix (**CE**).

Step 6: Map the equations to function. This mapping is obtained through matrix multiplication of Function-Component matrix and Component-Equation matrix. The information is recorded in a Function-Equation matrix (**FE**). The equation for this mapping is **FC·CE = FE**. Figure 2 illustrates this computation. Here, a 2x2 Function-Component matrix is multiplied with a 2x2 Component-Equation matrix resulting in a 2x2 Function-Equation matrix.

Step 7: Repeat the above steps for different products. Additional products are mined for Function-Component and Function-Equation matrices.

Step 8: Search for equation similarities. Review the recorded function-component and function-equation information and search for mathematical similarities. At this point, similarities are judgmentally observed. Adapting or developing strict metrics for mathematical similarity is beyond the scope of this article.

We now show how these eight steps are implemented, using a jigsaw and a palm sander as examples. We also seek to answer a key fundamental question: is there common functional and parametric model similarity? This question is at the heart of discovering basic knowledge about function form relationships and enabling quantitative analysis at the earliest stages of design synthesis.

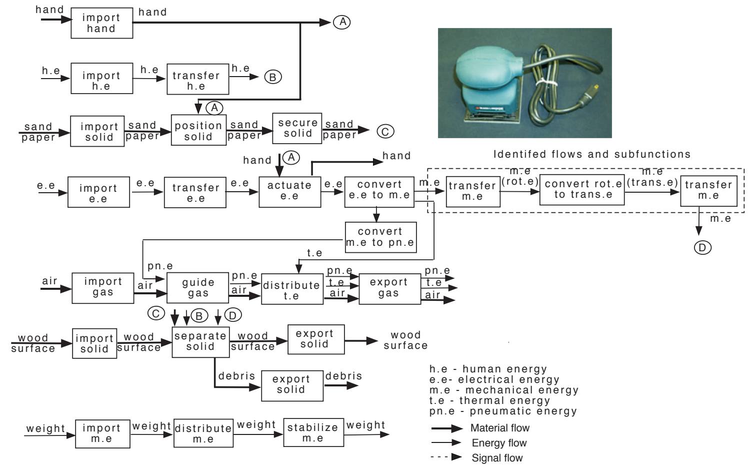


Figure 4. Functional model of a palm sander.

Completing step 1, a functional model of a jigsaw and a palm sander are presented in Figure 3 and Figure 4 respectively. Proceeding to step 2, the flow of mechanical energy and three functions operating on that flow are related to the customer need of longer stroke length for the jigsaw. Similarly, the flow of mechanical energy is the flow if interest for the palm sander. As described in step 3 above, three functions are decomposed and a more detailed functional model composed. This functional model is presented in Figure 5. Similarly for a palm sander, the flow of mechanical energy and three functions operating on that flow are identified. These functions are decomposed, modeled, and presented in Figure 6. To allow compact representation on the page, in these figures and figures and tables to follow, the following abbreviations have been used: m.e. for mechanical energy, rot. e. for rotational energy, and trans. e. for translational energy.

The fourth step involves mapping these functions to the parts described in the bill of materials of a jigsaw. The Function-Component matrix is developed for a jigsaw and is shown in Table

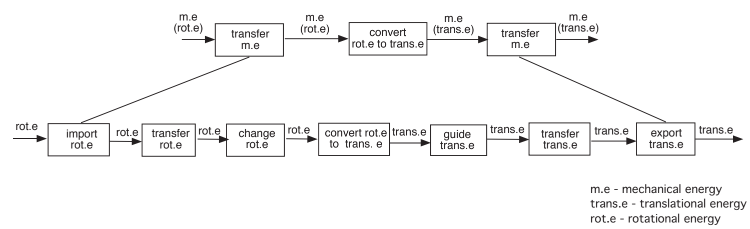


Figure 5. Decomposition of functions in a jigsaw.

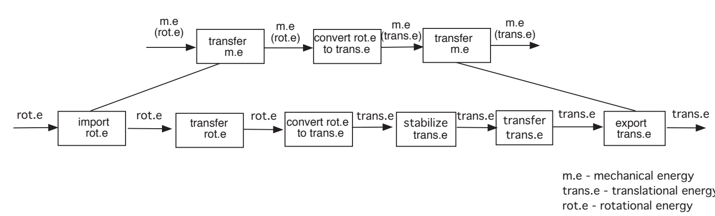


Figure 6. Decomposition of functions in a palm sander.

Table 1. Function Component Matrix of a Jigsaw

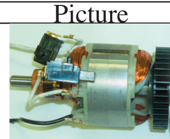
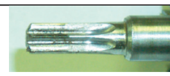
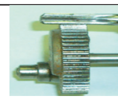
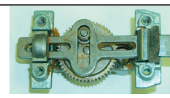
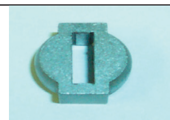
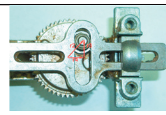
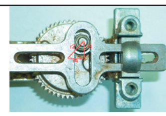
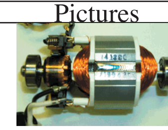
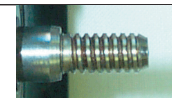
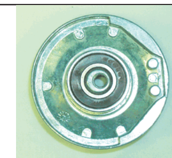
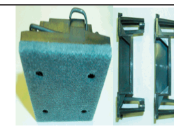
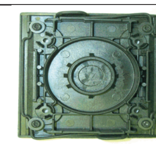
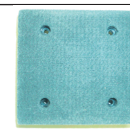
Function	Component	Picture
Import rot.e	Motor	
Transfer rot.e	Shaft	
Change rot.e	Gear train	
Convert rot.e to trans.e	Scotch yoke mechanism	
Guide trans.e	Vertical guides	
Transfer trans.e	Link length of the mechanism	
Export trans.e	Link length of the mechanism	

Table 2. Function Component Matrix of a Palm Sander

Function	Component	Pictures
Import rot.e	Motor	
Transfer rot.e	Shaft	
Convert rot.e to trans.e	Eccentric Cam	
Stabilize trans.e	Spring, damper	
Transfer trans.e	Sanding plate	
Export trans.e	Sand Paper	

1. Similarly, functions are mapped to parts described in the bill of materials of a palm sander. The Function-Component matrix is developed for a palm sander and is shown in Table 2.

In step 5, design equations are developed for different components of a jigsaw. These equations are shown in a Component-Equation matrix in Table 3. The design equations developed for different components of a palm sander are shown in a Component-Equation matrix in Table 4. The equations presented in these tables were derived from basic theory as presented in Shigley and Mischke, (2001) and Avallone and Baumeister, (1996). The equations used here are appropriate for many critical design decisions and specifications. In general, various levels of abstraction are used in design models to produce more or less accuracy and resolution in the results (for example, a lumped parameter model versus a continuous model of a vibrating beam). Selecting the most appropriate model for a design decision has been addressed to some extent in the literature (Doraiswamy 1999 and 2000; Radhakrishnan, 2000), though it remains a largely unsolved problem and is beyond the scope of this paper.

Step 6 is the matrix multiplication of Function-Component and Component-Equation matrices resulting in the Function-Equation matrix. Function-Equation matrices for the jigsaw and palm sander are shown in Table 5 and Table 6, respectively.

Step 7 is to repeat the process as more products are added to the knowledge base. The next step, step 8, is to explore the knowledge base for similarities in function and parametric model. This is done next is significant detail in the following section.

4. FUNCTION-EQUATION SIMILARITIES AND DIFFERENCES

Following procedure discussed in section 3, we have developed functional models and the associated design equations for a hand held jigsaw and a palm sander. These mathematical design equations were developed using basic engineering principles as applied to the specific components used in each product to solve the associated function. With the resultant initial knowledge base established, we now move to step 8 of the procedure: we attempt to extract general equations for functions that will describe the performance of many potential solutions (components). Reviewing Tables 5 and 6, one can observe that in some cases the same function with a different physical solution produced a different design equation. In other cases, the same function was solved by an almost identical component in both products resulting in the same design equations. A key finding in the initial comparison of functions and design equations is the similarity in the design equations for the function of convert rotational energy to translational energy in the different products. We will explore this similarity in detail in Section 4.2. First, a short overview of the results is presented.

4.1 Overview of Results of Step 8

With the basic steps of the research method completed, we are in a position to explore this created knowledge for similarities and differences in the mathematical equations that describe the same function in different products. These similarities and

Table 3. Component Equation Matrix of a Jigsaw

Component	Equations	
Motor	$\tau = A - B\omega$	(1)
Shaft	$\frac{n\sigma_a}{S_e} + \frac{n\sigma_m}{S_{ut}} = 1$	(2)
	$\frac{n\frac{16A}{\pi d^3}}{S_e} + \frac{n\frac{16B}{\pi d^3}}{S_{ut}} = 1$	(3)
	$d = \frac{16n}{d^3}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)^{1/3}$	(4)
	$\frac{1}{n} = \frac{16}{\pi d^3}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)$	(5)
	$\sigma_a = \frac{16A}{\pi d^3}$	(6)
	$\sigma_m = \frac{16B}{\pi d^3}$	(7)
	$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$	(8)
	$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$	(9)
Gear Train	$\frac{\omega_2}{\omega_1} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$	(10)
	$J_1\ddot{\theta}_1 + f_1\dot{\theta}_1 + \tau_1 = \tau_m$	(11)
	$J_2\ddot{\theta}_2 + f_2\dot{\theta}_2 + \tau_L = \tau_2$	(12)
	$T_1\omega_1 = T_2\omega_2$	(13)
	$T_1\theta_1 = T_2\theta_2$	(14)
Scotch Yoke Mechanism	$r_2 \sin\theta_2 = r_3$	(15)
	$r_2 \cos\theta_2 = r_4$	(16)
	$\frac{\partial r_3}{\partial t} = r_2 \cos\theta_2 \frac{\partial \theta_2}{\partial t}$	(17)
	$\frac{\partial r_4}{\partial t} = -r_2 \sin\theta_2 \frac{\partial \theta_2}{\partial t}$	(18)
	$\frac{\partial^2 r_3}{\partial t^2} = r_2 \cos\theta_2 \frac{\partial^2 \theta_2}{\partial t^2} - r_2 \sin\theta_2\left(\frac{\partial \theta_2}{\partial t}\right)^2$	(19)
	$\frac{\partial^2 r_4}{\partial t^2} = r_2 \sin\theta_2 \frac{\partial^2 \theta_2}{\partial t^2} + r_2 \cos\theta_2\left(\frac{\partial \theta_2}{\partial t}\right)^2$	(20)
Guides	$F_{14} = \frac{r_3}{r_4} F$	(21)
Link Length	$r_{out} = r_{in} + l$	(22)
Link Length	w, d	(23)

Table 4. Component Equation Matrix of a Palm Sander

Component	Equation	
Motor	$\tau = A - B\omega$	(24)
Shaft	$\frac{n\sigma_a}{S_e} + \frac{n\sigma_m}{S_{ut}} = 1$	(25)
	$\frac{n\frac{16A}{\pi d^3}}{S_e} + \frac{n\frac{16B}{\pi d^3}}{S_{ut}} = 1$	(26)
	$d = \frac{16n}{d^3}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)^{1/3}$	(27)
	$\frac{1}{n} = \frac{16}{\pi d^3}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)$	(28)
	$\sigma_a = \frac{16A}{\pi d^3}$	(29)
	$\sigma_m = \frac{16B}{\pi d^3}$	(30)
	$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$	(31)
	$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$	(32)
Eccentric Cam	$\frac{\partial x}{\partial t} = e\sin\theta \frac{\partial \theta}{\partial t}$	(33)
	$\frac{\partial^2 x}{\partial t^2} = e\sin\theta \frac{\partial^2 \theta}{\partial t^2} + e\cos\theta\left(\frac{\partial \theta}{\partial t}\right)^2$	(34)
	$F_r = m\frac{\partial^2 x}{\partial t^2}$	(35)
Spring,	$F_k = ke(1-\cos\theta)$	(36)
Damper	$F_c = ce\sin\theta \frac{\partial \theta}{\partial t}$	(37)
Sanding Plate	$F_{out}(x_2, y_2) \approx F_{in}(x_1, y_1)$	(38)
Sand Paper	l, w	(39)

differences are a mathematical link between the requirements of function, the synthesis of form, and the analysis of engineering design. Proceeding, we select the common functions from the jigsaw and palm sander and search for the critical similarities and differences between the products and associated function solutions. The comparison of the two products and the generalized equations are presented in Table 7. Table 7 presents the common functions along with their associated design equations and generalized equations for both the products. In this section we explore in detail the similarities and differences in these equations.

For importing rotational energy, a DC motor is used in both the products. This motor is governed by the torque-speed equation $\tau = A - B\omega$, which is a function of the motor characteristics where A and B are motor constants. This relationship, which is typical of some DC motors, indicates that the input torque varies with the crank speed. For the solution of transferring rotational energy, a shaft was used in both products. Thus equivalent design equations directly follow. The associated design equations are for stress and strength analysis, thus fatigue equations are used. The equations contain diameter of the shaft (d) and the factor of safety (n).

Making the comparison between the different solution methods and associated design equations for the function of converting rotational energy to translational energy yields an interesting and powerful result. A scotch yoke mechanism in a jigsaw converts rotational energy to translational energy and an eccentric cam in a palm sander converts rotational energy to translational energy. Though these are different mechanisms, they solve the same function and both the mechanisms have similar design equations. We will explore these two different mechanisms in detail in Section 4.2.

The function of transfer translational energy was performed using a link length (from the jigsaw blade) and a sand paper plate in a sander. Though these physical solutions appear in some sense similar, useful design equations for displacement and force display some important differences. In a jigsaw the displacement of the output link (r_4) is important and thus displacement equations are developed. A relationship of output displacement as a function of input displacement $r_{out} = r_4 + l_{blade}$ is derived and used. In case of a palm sander, the force relationship plays the prominent role rather than displacement. The force with which the sanding plate interacts the surface is important. Here the abrasion of the surface is more affected by the force applied rather than the displacement. An eccentric cam with eccentricity (e) and mass (m) rotating with a frequency ω causes a reaction force (F_r). Thus a force equation $F_{out}(x_2, y_2) \approx F_r(x_1, y_1)$ is considered.

The export translational energy function was solved using a link length in a jigsaw and sand paper in a sander. In a jigsaw, a

Table 5. Function Equation Matrix of a Jigsaw.

Sub-functions	Equations	
Import rot.e	$\tau = A - B\omega$	(1)
Transfer rot.e	$\frac{n\sigma_a}{S_e} + \frac{n\sigma_m}{S_{ut}} = 1$	(2)
	$\frac{n\frac{16A}{\pi d^3}}{S_e} + \frac{n\frac{16B}{\pi d^3}}{S_{ut}} = 1$	(3)
	$d = \frac{16n}{d^3}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)^{1/3}$	(4)
	$\frac{1}{n} = \frac{16}{\pi d^3}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)$	(5)
	$\sigma_a = \frac{16A}{\pi d^3}$	(6)
	$\sigma_m = \frac{16B}{\pi d^3}$	(7)
	$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$	(8)
	$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$	(9)
Change rot.e	$\frac{\omega_2}{\omega_1} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2}$	(10)
	$J_1\ddot{\theta}_1 + f_1\dot{\theta}_1 + \tau_1 = \tau_m$	(11)
	$J_2\ddot{\theta}_2 + f_2\dot{\theta}_2 + \tau_L = \tau_2$	(12)
	$T_1\omega_1 = T_2\omega_2$	(13)
	$T_1\theta_1 = T_2\theta_2$	(14)
Convert rot.e to trans.e	$r_2 \sin\theta_2 = r_3$	(15)
	$r_2 \cos\theta_2 = r_4$	(16)
	$\frac{\partial r_3}{\partial t} = r_2 \cos\theta_2 \frac{\partial \theta_2}{\partial t}$	(17)
	$\frac{\partial r_4}{\partial t} = -r_2 \sin\theta_2 \frac{\partial \theta_2}{\partial t}$	(18)
	$\frac{\partial^2 r_3}{\partial t^2} = r_2 \cos\theta_2 \frac{\partial^2 \theta_2}{\partial t^2} - r_2 \sin\theta_2 \left(\frac{\partial \theta_2}{\partial t}\right)^2$	(19)
	$\frac{\partial^2 r_4}{\partial t^2} = r_2 \sin\theta_2 \frac{\partial^2 \theta_2}{\partial t^2} + r_2 \cos\theta_2 \left(\frac{\partial \theta_2}{\partial t}\right)^2$	(20)
Guide trans.e	$F_{14} = \frac{r_3}{r_4} F$	(21)
Transfer trans.e	$r_{out} = r_{in} + l$	(22)
Export trans.e	w, d	(23)

Table 6. Function Equation Matrix for a Palm Sander.

Sub-function	Equation	
Import rot.e	$\tau = A - B\omega$	(24)
Transfer rot.e	$\frac{n\sigma_a}{S_e} + \frac{n\sigma_m}{S_{ut}} = 1$	(25)
	$\frac{n\frac{16A}{\pi d^3}}{S_e} + \frac{n\frac{16B}{\pi d^3}}{S_{ut}} = 1$	(26)
	$d = \frac{16n}{d^3}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)^{1/3}$	(27)
	$\frac{1}{n} = \frac{16}{\pi d^3}\left(\frac{A}{S_e} + \frac{B}{S_{ut}}\right)$	(28)
	$\sigma_a = \frac{16A}{\pi d^3}$	(29)
	$\sigma_m = \frac{16B}{\pi d^3}$	(30)
	$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$	(31)
	$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$	(32)
Convert rot.e to trans.e	$\frac{\partial x}{\partial t} = e \sin\theta \frac{\partial \theta}{\partial t}$	(33)
	$\frac{\partial^2 x}{\partial t^2} = e \sin\theta \frac{\partial^2 \theta}{\partial t^2} + e \cos\theta \left(\frac{\partial \theta}{\partial t}\right)^2$	(34)
	$F_r = m \frac{\partial^2 x}{\partial t^2}$	(35)
Stabilize trans.e	$F_k = ke(1 - \cos\theta)$	(36)
	$F_c = ce \sin\theta \frac{\partial \theta}{\partial t}$	(37)
Transfer trans.e	$F_{out}(x_2, y_2) \approx F_{in}(x_1, y_1)$	(38)
Export trans.e	l, w	(39)

blade that is connected to an output link exports the translational energy. In order for the blade to serve its purpose, a desirable width of a blade and the depth of a cut are important and thus width of the blade (w) and depth of cut (d) were developed. For the sander, the paper width (w) and length (l) serve better for design analysis. Because of the simplicity of the physical solutions for these functions, no models or equations are needed to set the design parameters.

4.2 Functional and Mathematical Equivalence: Jigsaw Scotch Yoke and Palm Sander Eccentric Cam

In case of the jigsaw and the palm sander, different physical solutions were used to solve the function of convert rotational energy to translational energy. Now we explore the key question: is there some similarity between the mathematical abstractions of two different form solutions, in this case a scotch yoke and an eccentric cam, used to solve the same function in different products. Functional models of the scotch yoke mechanism and eccentric cam have been developed and the equations for the corresponding functions developed. First, we search for kinematic equivalence, then dynamic equivalence.

4.2.1 Kinematic Equivalence

The scotch yoke mechanism and an eccentric cam have functional equivalence. Both mechanisms solve the same function of converting rotational energy to translational energy. We now explore the mathematical similarities between the two different mechanisms. The following figures and equations show how an eccentric cam with a follower face and a scotch yoke mechanism are kinematically equivalent. Both the scotch yoke mechanism and eccentric cam have relatively simple mathematical representations. An eccentric cam is circular in form, but the center of the circle O, is offset at a distance e from the center of the camshaft P. The scotch yoke mechanism is kinematically equivalent to the eccentric cam.

Looking at equations for kinematic equivalence derived following the basic principles of machine dynamics, we can conclude that the mathematical representations of linear acceleration of both an eccentric cam (considering only one dimension) and a scotch yoke to be similar. Particularly, we can relate that the radius of the scotch yoke mechanism and the eccentricity of the cam to be directly proportional. The acceleration for a jigsaw and a palm sander are given by:

$$a = R(\sin\theta\alpha + \cos\theta\omega^2) \quad (1)$$

and

$$a = R(\sin\theta\alpha + \cos\theta\omega^2). \quad (2)$$

Table 7. Comparison between Jigsaw and Palm sander

Sub-function	Jigsaw	Palm sander	General Equations
Import rot.e	$\tau = A - B\omega$	$\tau = A - B\omega$	$\tau = A - B\omega$
Transfer rot.e	$d = \frac{16n}{d^3}\left(\frac{A}{S_e}+\frac{B}{S_{ut}}\right)^{1/3}$ $\frac{1}{n}=\frac{16}{\pi d^3}\left(\frac{A}{S_e}+\frac{B}{S_{ut}}\right)$	$d = \frac{16n}{d^3}\left(\frac{A}{S_e}+\frac{B}{S_{ut}}\right)^{1/3}$ $\frac{1}{n}=\frac{16}{\pi d^3}\left(\frac{A}{S_e}+\frac{B}{S_{ut}}\right)$	$d = \frac{16n}{d^3}\left(\frac{A}{S_e}+\frac{B}{S_{ut}}\right)^{1/3}$ $\frac{1}{n}=\frac{16}{\pi d^3}\left(\frac{A}{S_e}+\frac{B}{S_{ut}}\right)$
Convert rot.e to trans. e	$\frac{\partial r_4}{\partial t} = -r_2 \sin\theta_2 \frac{\partial \theta_2}{\partial t}$ $\frac{\partial^2 r_4}{\partial t^2} = r_2 \sin\theta_2 \frac{\partial^2 \theta_2}{\partial t^2} + r_2 \cos\theta_2 \left(\frac{\partial \theta_2}{\partial t}\right)^2$	$\frac{\partial x}{\partial t} = e \sin\theta \frac{\partial \theta}{\partial t}$ $\frac{\partial^2 x}{\partial t^2} = e \sin\theta \frac{\partial^2 \theta}{\partial t^2} + e \cos\theta \left(\frac{\partial \theta}{\partial t}\right)^2$ $F_r = m\frac{\partial^2 x}{\partial t^2}$	$v = \frac{\partial R}{\partial t} = k \sin\theta \frac{\partial \theta}{\partial t}$ $\frac{\partial^2 R}{\partial t^2} = k \sin\theta \frac{\partial^2 \theta}{\partial t^2} +$ $k \cos\theta \left(\frac{\partial \theta}{\partial t}\right)^2$ $F_r = m\frac{\partial^2 x}{\partial t^2}$
Transfer trans.e	$r_{out} = r_4 + l_{blade}$	$F_{out}(x_2, y_2) \approx F_r(x_1, y_1)$	$r_{out} = R + l$ $F_{out} \approx F_R$
Export trans.e	w, d	l, w	Geometric constraints

From these two equations we see that R from the scotch yoke mechanism is similar to e from an eccentric cam in particular $R \propto e$.

These equations for acceleration can be generalized to

$$a = k\left(\sin\theta\alpha + \cos\theta\omega^2\right), \tag{3}$$

where k is equivalent to either R or e depending upon the mechanism. This result leads to the following conclusion: a class of solutions for converting rotational energy to translational energy will have as a relevant design equation, an equation of the form given by Equation (3). This generalized equation shows a relation between translational acceleration as a function of angular acceleration (α) and angular velocity (ω). This equation was developed from positional and velocity equations. In order to visualize how these equations relate to the rotational and translational motion we present a more generalized positional and velocity equations. The generalized equations $x = k(1-\cos\theta)$ and $v = k\sin\theta\omega$ show a direct relationship between rotational and translational motion. Here $x = f(\theta)$ and $v = f(\omega)$ represent linear position and linear velocity as a function rotational components. In the case of a jigsaw and a palm sander, acceleration of the output link is important and thus acceleration equations were used for showing kinematic equivalence.

4.2.2 Dynamic Equivalence

Dynamic motion analysis is performed on slider crank mechanism. It contains free body diagrams of the three moving links, including inertia forces and torques. Similarly dynamic motion analysis is performed on an eccentric cam. Free body diagrams and its associated forces were used in the analysis. Here we present the equations of motion of a slider crank when driven by a motor. The final equation is presented below,

$$\left[I + mr^2(\sin\theta)(\sin\theta)\right]\frac{\partial^2\theta}{\partial t^2} + B\frac{\partial\theta}{\partial t} - A + mr^2\omega^2\cos\theta\sin\theta = 0. \tag{4}$$

We present the equations of motion of an eccentric cam when driven by a motor. The final equation is shown below,

$$\left[I + me^2\sin^2\theta\right]\frac{\partial^2\theta}{\partial t^2} + \left[Ce^2\sin^2\theta + B\right]\frac{\partial\theta}{\partial t} + \left[me^2\omega^2\cos\theta\sin\theta + ke^2\sin\theta(1-\cos\theta) - A\right] = 0. \tag{5}$$

Comparing Equations (4) and (5), we see that these mathematical representations are similar except for two terms that result due to the presence of a spring and a damper in an eccentric cam. In order to model the system correctly, the spring and damper were not excluded from the analysis. From the specific context of designing this eccentric mechanism for the sander, the dynamic effects caused by the spring and damper are important. The spring and damper aspects contribute to the motion of the sandpaper across the solid that is being sanded. In order to visualize equations without the existence of a spring and a damper, we can eliminate the terms using these by substituting $k, c \to 0$. Having done this, Equations (4) and (5) are of the same form. Equation (5) can be thought as a more generic equation for the dynamic representation of convert rotational to translational motion function. From the general equation, various classes of form specific solutions can be obtained based on certain conditions. One such condition would be the elimination of damping and compliance terms. In order to see this, we examine the dynamic equation for a scotch yoke mechanism that does not have damping and compliance components. Thus we eliminate the damping and compliance terms $k, c \to 0$ in Equation (5) and Equation (4) is obtained. Consequently, Equation (4) can be thought as a special case of Equation (5). From this, we can now say that different form solutions can be obtained by different simplifications of the generic equation we derived for the function convert rotational

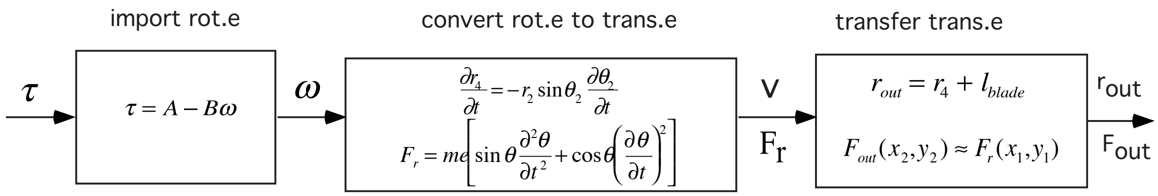


Figure 7. Parametric function chain model for sub-functions: import rot.e, convert rot.e to trans.e to transfer trans.e

to translational motion.

4.2.3 Parametric Function Chain Models

At the earliest stages of design, as mentioned in Figure 1, the ability to produce useful design models decreases as one continues to move upwards on the right side of Figure 1. Using a functional based approach to design synthesis, the designer can develop sub-system level parametric design models from information contained in the functional model. Also, as discussed above, at some level of modeling abstraction these design models can be directly linked to function without explicit representation of form. Thus, a designer can begin to explore quantitative estimates of product performance based on a functional model of the product. As specific performance emphases are identified, suitable form solutions can be identified, and their design equations substituted into the sub-system level design models and performance estimates refined. In the context of the sander and jigsaw, we present an example function chain that imports rotational energy, converts rotational energy to translational energy and transfers translational energy.

The mathematical equations from the function chain for a jigsaw are considered first. Our effort is to establish a relationship $r_{out} = f(\tau)$ that tells us how the input and output parameter are related. In order to construct this relation we need to select the appropriate terms in the equations presented in Figure 7. Rearranging the terms in the equations presented in Figure 7, we get the following equations

$$\omega = \frac{A - \tau}{B}, \tag{6}$$

$$\frac{\partial r_4}{\partial t} = -r_2 \sin\theta_2 \frac{\partial\theta_2}{\partial t}, \tag{7}$$

and

$$r_{out} = r_4 + l_{blade} . \tag{8}$$

Using the relation $\omega = \frac{\partial\theta_2}{\partial t}$, and Equation (6) in Equation (7) we get

$$\frac{\partial r_4}{\partial t} = -r_2 \sin\theta_2 \left(\frac{A - \tau}{B}\right). \tag{9}$$

Integrating Equation (9) we get $r_4 = \int -r_2 \sin\theta_2 \left(\frac{A-\tau}{B}\right) dt$. Substituting this relation in Equation (8) gives an overall equation describing the performance parameter of displacement of the output link as:

$$r_{out} = l_{blade} + \int -r_2 \sin\theta_2 \left(\frac{A - \tau}{B}\right) dt . \tag{10}$$

This equation encompasses various design variables driver link (r_2), Motor constants A and B and finally input torque (τ). Thus this equation parameterizes the design problem and allows the designer to determine a set of acceptable design parameters to achieve the desired performance.

Now we consider the mathematical equations from the function chain for a palm sander. Rearranging the terms in the equations presented in Figure 7, we obtain the following equations:

$$\omega = \frac{A - \tau}{B}, \tag{11}$$

$$F_r = m \frac{\partial^2 x}{\partial t^2}, \tag{12}$$

and

$$F_{out}(x_2, y_2) \approx F_r(x_1, y_1) . \tag{13}$$

Using Equations (11) & (12) in Equation (13), we establish an overall equation describing the performance parameter of output force as:

$$F_{out}(x_2, y_2) = me\left[\sin\theta \frac{\partial}{\partial t}\left(\frac{A - \tau}{B}\right) + \cos\theta \left(\frac{A - \tau}{B}\right)^2\right]. \tag{14}$$

This relationship between $F_{out} = f(\tau)$ parameterizes the design problem. The equation contains different design parameters such as the mass of the eccentric cam (m) eccentricity (e) and motor constants A and B.

These equation models make it easy to quickly see the interdependency of design parameters and try out various combinations of parameters to determine the best overall design. While the equations are derived from two specific products, the individual functions can be generalized and applied to any product with that functionality. Of critical importance to design is that these design equations were developed in parallel with functional descriptions of the product. By employing this equation development method, the designer brings analysis a step closer to synthesis.

5. SUMMARY AND CONCLUSIONS

Using the research method, we were able to conclude that a link between synthesis and associated analysis does exist. We explored this relationship and have presented the results in Figure 8 and Figure 9. Through the research presented here, we have shown how a designer can traverse the path between synthesis and associated analysis once a functional model of a product is derived. For the examples of the jigsaw shown in Figure 8 and the palm sander shown in Figure 9, the designer can traverse the path between synthesis and analysis one step earlier (or one level higher with respect to Figure 1) in the design stage. Also,

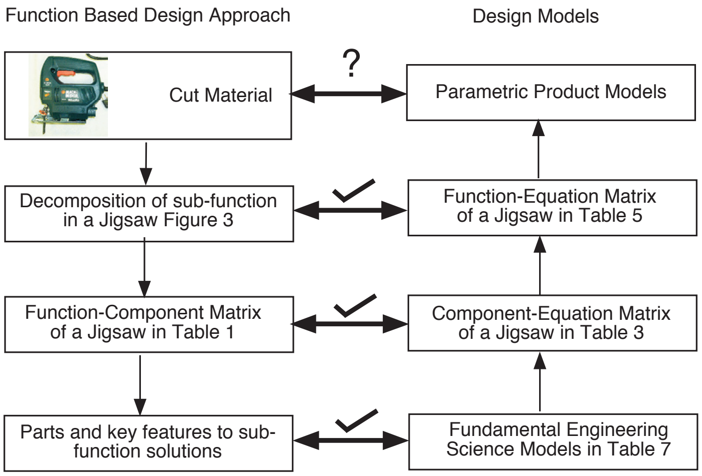


Figure 8. Functional Approach to a Jigsaw.

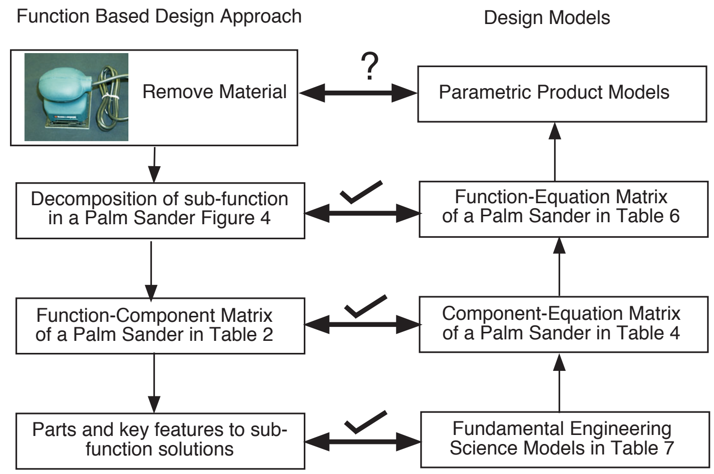


Figure 9. Functional Approach to a Palm Sander.

parametric equations can be developed in conjunction with the functional model that represents a range of specific form solutions. Thus, the designer can develop performance estimates earlier in the design process: we are moving toward simultaneous synthesis and analysis.

Toward developing full product system models from functional models, we have shown the first step of combining the equations associated with individual functions into sub-system parametric models for a function chain. These types of models allow a designer to explore the impact of different design parameters on the performance output of a particular function chain. With the research presented in this article, we fall short of creating a complete system model of a product from a black box functional description. However, we are a step closer and have some promising results that indicate that a framework for such product models can be developed. Before establishing that such a framework can be created, more fundamental issues, such as what constitutes a high-level performance model, must be addressed first.

This paper has described a new approach that enables parametric design models to be constructed during the early stages of conceptual design. Work has been presented on different stages of design development and synthesis during conceptual phase of design process. We have shown that useful mathematical models can be derived from functions in a functional model. These mathematical models were used to compare differing solution principles in concept variants based on their common functionality. Here we have shown how a scotch yoke mechanism in a jigsaw is similar to an eccentric cam in a palm sander. We have shown that a common equation or abstracted equation can be derived for a given secondary level functions. These secondary level functions may provide sufficient information to map performance equations to design parameters. At a conceptual design stage, we can produce system level performance equations to evaluate concept variants and initiate embodiment design phase. These mathematical models can be used at all stages of product development process, from concept evaluation, product testing, analysis and optimization of parameters. Future work will require modeling of systems in multiple physical domains such as electrical, fluids, kinematics and pneumatics.

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